

A Construction of Non-Kähler Calabi-Yau Manifolds and New Solutions to the Strominger System

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1 Introduction

This paper is a follow-up of an observation made in [Fei15b]. Nevertheless, I would like to say a few words about the general background first.

We call a compact complex manifold a Calabi-Yau, to be Kähler or not, if its canonical bundle is holomorphically trivial. Calabi-Yau manifolds, especially Calabi-Yau 3-folds, have been extensively studied ever since Yau's solution to the Calabi conjecture [Yau77, Yau78b]. Among many other enthralling problems, motivated from either geometry or string theory, there stands out the moduli problem. There exists a vast collection of literature devoted to understanding the moduli space of Calabi-Yau manifolds. We refer to the beautiful survey paper [Yau09] and the references therein for more details. In our context, we would like to single out what is usually called the “Reid's fantasy”. In [Rei87], Reid made the wild conjecture that all the reasonably nice compact 3-folds with trivial canonical bundle (including non-Kähler ones necessarily) can be connected with each other via conifold transitions. Reid's idea was supported by the work of Candelas-de la Ossa [CdIO90], where explicit Ricci-flat Kähler metrics are found on both deformed and resolved conifolds and their limiting behaviors are analyzed.

In order to globalize Candelas-de la Ossa's result, it is natural to ask the question that how canonical we can choose Hermitian metrics on compact non-Kähler Calabi-Yau manifolds. The first way to deal with this problem is to understand how to generalize the Calabi-Yau theorem in the non-Kähler setting. In particular, the balanced version of Gauduchon conjecture remains to be solved. A consequence of this conjecture is that on complex balanced manifolds with trivial canonical bundle one can always find a balanced metric within a suitable cohomology class such that its first Bott-Chern form vanishes. Progress was made by Székelyhidi-Tosatti-Weinkove [STW15] in this direction.

A second approach to this problem is to solve the Strominger system. This is a system of PDEs proposed by Strominger [Str86] in the study of heterotic strings with torsion which we shall formulate. Let (X^3, g, J) be an Hermitian 3-fold (not necessarily Kähler) with holomorphically trivial canonical bundle and let Ω be a nowhere-vanishing holomorphic $(3, 0)$ -form on X . We denote the positive $(1, 1)$ -form associated with g by ω and the curvature form of $(T_{\mathbb{C}}X, g)$ by R . In addition, let (E, h) be a holomorphic vector bundle with metric over X and let F be its curvature form. The Strominger system consists of the following equations:

$$\begin{aligned} (1) \quad & F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0, \\ (2) \quad & \sqrt{-1} \partial \bar{\partial} \omega = \frac{\alpha'}{4} (\text{Tr}(R \wedge R) - \text{Tr}(F \wedge F)), \\ (3) \quad & d(\|\Omega\|_{\omega} \cdot \omega^2) = 0. \end{aligned}$$

Equations (1), (2) and (3) are known as the Hermitian-Yang-Mills equation, the anomaly cancellation equation and the conformally balanced equation respectively.

If we assume that ω is Kähler and take $E = T_{\mathbb{C}}X$, the anomaly cancellation (2) is automatic. It follows that the whole system is reduced to an equation requiring g to be Ricci-flat. In this sense,

the Strominger system generalizes the complex Monge-Ampère equation used in Kähler geometry. Therefore we may regard solutions to Strominger system as canonical metrics, even on a non-Kähler background. Actually, the first method mentioned above can be incorporated into this picture. It is well-known that for X with trivial canonical bundle, the conformally balanced condition (3) is equivalent to that the restricted holonomy of X with respect to the Strominger-Bismut connection is contained in $SU(3)$, see [Str86, Section 2] and [LY05, Lemma 3.1]. However, this is very far from requiring Ω to be parallel under the Strominger-Bismut connection. It turns out the latter condition is equivalent to that $\|\Omega\|_\omega$ is a constant and ω is a balanced metric, i.e., solving the balanced version of Gauduchon conjecture.

The Strominger system is very hard to solve in general. The reason lies in the fact that the anomaly cancellation equation (2) is an equation of 4-forms. After certain perturbative solutions described in [Str86], the first smooth irreducible solutions for $U(4)$ or $U(5)$ principal bundles over Kähler Calabi-Yau's were due to Li and Yau [LY05]. Later, a set of genuine non-Kähler solutions were obtained by Fu and Yau [FY08]. A great deal of work has been done in recent years, see [FIUV09], [FTY09], [AGF12], [AGS14], [dlOS14], [FY15], [GFRT15] and the references therein.

Following [Mic82], we say an Hermitian metric ω on a complex n -fold is *balanced* if $d(\omega^{n-1}) = 0$. As a consequence of (3), X must support a balanced metric. Thanks to the work of [Cle83], [Fri86] and [FLY12], there are lots of non-Kähler Calabi-Yau 3-folds with balanced metric which can be obtained by taking a sequence of conifold transitions starting from a projective Calabi-Yau 3-fold. Other techniques like branched double covering [LWY14] may also be useful and we refer to the survey paper [Tos15] for more constructions. However, it seems that for most of these constructions, the Strominger system is way too hard to attack. As far as the author knows, the only successes were made on those T^2 -bundles over K3 surfaces constructed by [GP04] and quotients of various Lie groups over lattices.

This paper is divided into two parts. In Part I, following the observation made in [Fei15b], we provide a new way to find non-Kähler manifolds which generalizes the classical construction of Calabi [Cal58] and Gray [Gra69]. In particular, we obtain a series of non-Kähler Calabi-Yau 3-folds with natural balanced metric. These 3-folds are holomorphic fiber bundles over Riemann surfaces of genus $g \geq 3$ with hyperkähler fibers. For Part II, we make use of this fibration structure to write down a suitable ansatz for the Strominger system. By solving an algebraic equation, we obtain explicit degenerate solutions to the Strominger system with $F = 0$. Along the way we also prove that our models do not admit any pluriclosed metrics, answering a question of Fu-Wang-Wu [FWW13, Section 1]. Finally we make a discussion about the geometry of degeneracy loci.

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Part I

Geometric Construction

2 The Classical Construction of Calabi and Gray

For any oriented immersed hypersurface X in \mathbb{R}^7 , Calabi [Cal58] discovered that X automatically admits an almost complex structure J_0 from the following construction. Let us identify \mathbb{R}^7 with $\text{Im}(\mathbb{O})$, the space of purely imaginary octonions. There is a cross product \times defined on $\text{Im}(\mathbb{O})$ which

can be expressed by

$$a \times b = a \cdot b + \langle a, b \rangle,$$

where \cdot is the octonion multiplication and $\langle -, - \rangle$ is the standard inner product on \mathbb{O} . For any point $p \in X$, the almost complex structure $(J_0)_p : T_p X \rightarrow T_p X$ is defined by

$$(J_0)_p(v) = \nu_p \times v, \quad \forall v \in T_p X,$$

where ν_p is the unit positive normal of M at p with respect to the standard metric and orientation. Using properties of cross product on $\text{Im}(\mathbb{O})$, it is not hard to check that $J_0^2 = -\text{id}$ and we get an almost complex structure.

Calabi also derived the condition for J_0 to be integrable. In particular, he proved that if Σ is an oriented immersed minimal surface in \mathbb{R}^3 , then the almost complex structure induced on $X = \Sigma \times \mathbb{R}^4 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^4 = \text{Im}(\mathbb{O})$ is integrable. Using such construction, he gave the first example that the Chern classes of a complex manifold is not determined by its underlying smooth structure, answering a question of Hirzebruch.

Calabi's construction was later generalized by Gray [Gra69] to manifolds with vector cross product. That is, one can replace \mathbb{R}^7 by any manifolds with a G_2 -structure. In particular, if Σ_g is an oriented minimal surface of genus g in flat T^3 and M is a compact hyperkähler 4-manifold (either T^4 or a K3 surface), then the induced J_0 on $X = \Sigma_g \times M \hookrightarrow T^3 \times M$ is integrable. Notice that though diffeomorphically X is a product, the holomorphic structure on X is twisted, as long as Σ_g is not totally geodesic, i.e., a flat subtorus. Such minimal surfaces Σ_g exist for and only for every $g \geq 3$, see [MI90, Corollary 3.1, Corollary 10.1] and [Tra08, Theorem 1].

It was further observed in [Fei15b, Theorem 5.1] that these X are actually compact non-Kähler Calabi-Yau's. The argument goes as follows. As the fundamental 3-form φ of the G_2 -manifold $T^3 \times M$ is closed, its restriction on X is also closed. On the other hand, the restricted 3-form is the real part of a complex (3,0)-form Ω . Using the fact J_0 is integrable, we conclude that Ω is holomorphic and therefore trivializes the canonical bundle of X .

There is also a natural induced metric on X from the ambient G_2 -manifold $T^3 \times M$. It is classically known [Gra69] that this metric ω is Hermitian and balanced. In addition, $\|\Omega\|_\omega$ is of constant length and therefore Ω is parallel under both the Chern connection and the Strominger-Bismut connection. Thus (X, ω) is a *special balanced 3-fold*, using the terminology from [Fei15b].

Moreover, the fibration

$$\pi : (\Sigma_g \times M, J_0) \rightarrow \Sigma_g$$

is holomorphic with holomorphic sections of the form $\Sigma_g \times \{\text{pt}\}$.

3 Explicit Calculation

In this section, we lay the foundation for future calculations.

Let us first write down the complex structure J_0 explicitly. Let e_1, e_2, e_3 be an orthonormal basis of parallel vector fields on T^3 and let e^1, e^2, e^3 be the dual 1-forms. Fix I, J, K a set of pairwise anti-commuting complex structures on the hyperkähler manifold M , and denote the associated Kähler forms by ω_I, ω_J and ω_K respectively. Let $G : \Sigma_g \rightarrow S^2 \subset \mathbb{R}^3$ be the Gauss map and write its components as

$$G(z) = (\alpha(z), \beta(z), \gamma(z)) \in \mathbb{R}^3, \quad \forall z \in \Sigma_g.$$

Thus

$$\nu(z) = \alpha(z)e_1 + \beta(z)e_2 + \gamma(z)e_3.$$

Notice that the cross product on $\Sigma_g \times M$ is determined by the fundamental 3-form

$$\varphi = e^1 \wedge \omega_I + e^2 \wedge \omega_J + e^3 \wedge \omega_K - e^1 \wedge e^2 \wedge e^3.^1$$

¹It should be noted that the orientation induced on T^3 differs from the standard one by a sign.

It is not hard to see that

$$\begin{aligned}
(4) \quad J_0 e_1 &= -\gamma e_2 + \beta e_3, \\
J_0 e_2 &= \gamma e_1 - \alpha e_3, \\
J_0 e_3 &= -\beta e_1 + \alpha e_2, \\
J_0 v &= \alpha I v + \beta J v + \gamma K v,
\end{aligned}$$

for arbitrary vector field v tangent to fibers of $\pi : X \rightarrow \Sigma_g$.

The action of J_0 on forms can be obtained easily as follow

$$\begin{aligned}
J_0 e^1 &= \gamma e^2 - \beta e^3, \\
J_0 e^2 &= -\gamma e^1 + \alpha e^3, \\
J_0 e^3 &= \beta e^1 - \alpha e^2, \\
J_0 \omega_I &= (2\alpha^2 - 1)\omega_I + 2\alpha\beta\omega_J + 2\alpha\gamma\omega_K, \\
J_0 \omega_J &= 2\alpha\beta\omega_I + (2\beta^2 - 1)\omega_J + 2\beta\gamma\omega_K, \\
J_0 \omega_K &= 2\alpha\gamma\omega_I + 2\beta\gamma\omega_J + (2\gamma^2 - 1)\omega_K.
\end{aligned}$$

Denote by ω_0 the induced metric on X from $T^3 \times M$, clearly,

$$(5) \quad \omega_0 = \omega + \alpha\omega_I + \beta\omega_J + \gamma\omega_K,$$

where ω is the induced Kähler metric on Σ_g .

Up to now we have not used the fact that Σ_g is minimal. Let $f : D \rightarrow \Sigma_g \subset \mathbb{R}^3$ given by

$$(u, v) \mapsto (f_1(u, v), f_2(u, v), f_3(u, v))$$

be an isothermal parametrization of Σ_g compatible with its orientation. Let $z = u + iv$ and

$$\varphi_j = \frac{\partial f_j}{\partial u} - i \frac{\partial f_j}{\partial v}$$

for $j = 1, 2, 3$. It is a well-known fact that Σ_g is a minimal surface is equivalent to that φ_j are holomorphic functions and

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0.$$

Setting

$$2\lambda = \varphi_1 \bar{\varphi}_1 + \varphi_2 \bar{\varphi}_2 + \varphi_3 \bar{\varphi}_3,$$

we can easily express α, β, γ as

$$\begin{aligned}
-2i\lambda\alpha &= \varphi_2 \bar{\varphi}_3 - \varphi_3 \bar{\varphi}_2, \\
-2i\lambda\beta &= \varphi_3 \bar{\varphi}_1 - \varphi_1 \bar{\varphi}_3, \\
-2i\lambda\gamma &= \varphi_1 \bar{\varphi}_2 - \varphi_2 \bar{\varphi}_1.
\end{aligned}$$

Without too much effort, one can check that

$$\varphi_1^{-1} \frac{\partial \alpha}{\partial \bar{z}} = \varphi_2^{-1} \frac{\partial \beta}{\partial \bar{z}} = \varphi_3^{-1} \frac{\partial \gamma}{\partial \bar{z}}.$$

We also have the relations

$$\begin{aligned}
(6) \quad -i \frac{\partial \alpha}{\partial \bar{z}} &= \beta \frac{\partial \gamma}{\partial \bar{z}} - \gamma \frac{\partial \beta}{\partial \bar{z}} \\
-i \frac{\partial \beta}{\partial \bar{z}} &= \gamma \frac{\partial \alpha}{\partial \bar{z}} - \alpha \frac{\partial \gamma}{\partial \bar{z}}, \\
-i \frac{\partial \gamma}{\partial \bar{z}} &= \alpha \frac{\partial \beta}{\partial \bar{z}} - \beta \frac{\partial \alpha}{\partial \bar{z}}.
\end{aligned}$$

4 Generalizations

In this section, we will generalize the classical construction of Calabi and Gray. A first observation is that the recipe (4) used for producing the complex structure J_0 make sense for any hypercomplex manifold M . More precisely, let N be a hypercomplex manifold of complex dimension $2n$, i.e., a smooth manifold of real dimension $4n$ endowed with three integrable complex structures I, J and K satisfying

$$I^2 = J^2 = K^2 = IJK = -\text{id}.$$

Let Σ_g be an oriented minimal surface immersed in T^3 as before, then we can define an almost complex structure J_0 on $\Sigma_g \times N$ using exactly the formula (4). A straightforward calculation of Nijenhuis tensor shows that J_0 is integrable, due to the relations we derived in (6). If we further assume that N is actually hyperkähler, then the associated Hermitian forms ω_I, ω_J and ω_K are closed. Notice that the naturally induced metric ω_0 on $\Sigma_g \times N$ is still of the form (5). We see immediately that

$$\omega_0^{2n} = (\alpha\omega_I + \beta\omega_J + \gamma\omega_K)^{2n} + 2n \cdot \omega \wedge (\alpha\omega_I + \beta\omega_J + \gamma\omega_K)^{2n-1}$$

is d-closed, hence ω_0 is a balanced metric.

Remark 4.1. Hypercomplex manifolds form a strictly larger category than hyperkähler manifolds. For instance, in real dimension 4, besides T^4 and K3 surfaces, we should also include the Hopf surfaces into the hypercomplex list, see [Boy88].

Our next observation relates the above construction to the twistor space of hypercomplex manifolds. Recall from [HKLR87, Section 3(F)] and [Kal98, Section 1] that the twistor space of a hypercomplex manifold N is constructed as follows. Parameterize $S^2 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}$ by $\zeta \in \mathbb{CP}^1$ via

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \bar{\zeta}}{1 + |\zeta|^2}, \frac{i(\bar{\zeta} - \zeta)}{1 + |\zeta|^2} \right).^2$$

The twistor space Z of N is defined to be the manifold $Z = \mathbb{CP}^1 \times N$ with the almost complex structure \mathfrak{J} given by

$$(7) \quad \mathfrak{J} = j \oplus (\alpha I_x + \beta J_x + \gamma K_x)$$

at point $(\zeta, x) \in \mathbb{CP}^1 \times N$, where j is the standard complex structure on \mathbb{CP}^1 with holomorphic coordinate ζ . It is a theorem of [HKLR87] and [Kal98] that \mathfrak{J} is integrable and the projection $p : Z \rightarrow \mathbb{CP}^1$ is a holomorphic fibration.

Recall that for an oriented minimal surface Σ_g in T^3 , the Gauss map $G : \Sigma_g \rightarrow \mathbb{CP}^1$ written in above coordinate

$$z \mapsto \zeta(z)$$

is holomorphic. Comparing the definition of J_0 (4) and \mathfrak{J} (7), we conclude immediately that J_0 is exactly the pullback of \mathfrak{J} by the Gauss map G . More precisely, we have the pullback square

$$\begin{array}{ccc} (\Sigma_g \times N, J_0) & \cong & G^*Z \xrightarrow{\tilde{G}} (Z, \mathfrak{J}) \\ \pi \downarrow & & \downarrow p \\ \Sigma_g & \xrightarrow{G} & \mathbb{CP}^1 \end{array}$$

All the maps in this diagram are holomorphic.

²The γ component here differs from [HKLR87] by a sign. If we follow the convention that $IJ = K$ on vector fields, then $IJ = -K$ on 1-forms. With this understood, one detects a sign issue in [HKLR87]'s calculation. It turns out our formula above gives the right complex structure.

This is a very useful viewpoint in practice. For instance, it allows us to compute the first Chern class of $(\Sigma_g \times N, J_0)$. Let $F_\pi = \ker d\pi$ and $F_p = \ker dp$ respectively, and we have two short exact sequences of bundles

$$\begin{aligned} 0 \rightarrow F_\pi \rightarrow TG^*Z \xrightarrow{d\pi} \pi^*T\Sigma_g \rightarrow 0, \\ 0 \rightarrow F_p \rightarrow TZ \xrightarrow{dp} p^*T\mathbb{CP}^1 \rightarrow 0. \end{aligned}$$

From the pullback square, we see that

$$F_\pi = \tilde{G}^*F_p,$$

therefore we conclude that

$$c_1(\Sigma_g \times N) = c_1(F_\pi) + \pi^*c_1(\Sigma_g) = \pi^*c_1(\Sigma_g) + \tilde{G}^*c_1(F_p).$$

It was proved in [HKLR87, Theorem 3.3] that $\wedge^2 F_p^* \otimes p^*\mathcal{O}(2)$ has a section which defines a holomorphic symplectic form on each fiber of p .³ It follows that

$$c_1(F_p) = p^*c_1(\mathcal{O}(2n)).$$

Let μ be the positive generator of $H^2(\Sigma_g, \mathbb{Z})$ and identify it with $\mu \otimes 1 \in H^*(\Sigma_g \times N, \mathbb{Z})$, we conclude that

$$c_1(\Sigma_g \times N) = (g-1)(2n-2)\mu,$$

where we have used the simple fact that $\deg G = g-1$ as a consequence of Gauss-Bonnet.

We see immediately that when $n = 1$, i.e., N is a hypercomplex 4-manifold, then $c_1(\Sigma_g \times N) = 0$. This is consistent with our result mentioned in Section 2. We also conclude that $c_1(\Sigma_g \times N)$ can always be represented by a nonnegative class, which is good enough for us to prove the following theorem.

Theorem 4.2. $(\Sigma_g \times N, J_0)$ does not support any Kähler metric.

To prove this theorem, we need to use Yau's generalized Schwarz lemma [Yau78a], which says

Theorem 4.3. (Yau, [Yau78a, Theorem 2])

Let P_1 be a complete Kähler manifold with nonnegative Ricci curvature and let P_2 be an Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant. Then any holomorphic map from P_1 to P_2 must be a constant.

Now we proceed to prove Theorem 4.2

Proof. Assume that $\Sigma_g \times N$ admits a Kähler metric, then by Yau's solution of Calabi conjecture [Yau77, Yau78b], we may choose the Kähler metric to have nonnegative Ricci curvature since $c_1(\Sigma_g \times N)$ is the Ricci form. On the other hand, as $g \geq 3$, we know that Σ_g admits a hyperbolic metric. This contradicts with Yau's Schwarz lemma since $\pi : \Sigma_g \times N \rightarrow \Sigma_g$ is not a constant. \square

Remark 4.4. Yau's Schwarz lemma was later generalized by Tosatti [Tos07] to the case that P_1 is a complete almost Hermitian manifold. In this case, one should use the 2nd Ricci curvature with respect to the so-called canonical connection, which coincides with the Chern connection in the integrable case. Our example of non-Kähler Calabi-Yau shows that one cannot replace the 2nd Ricci curvature by the 1st Ricci curvature.

We next prove that

Theorem 4.5. $\Sigma_g \times N$ admits balanced metrics.

³As long as we are only concerned with topology instead of holomorphic structure, this result is available to hypercomplex manifold as well.

Proof. An explicit balanced metric has been constructed at the beginning of this section in the case that N is hyperkähler. However, we will present a proof that is also applicable to the hypercomplex setting.

In [Tom15], Tomberg proved that Z admits a balanced metric. If we pull-back this form back to $\Sigma_g \times N$ via G , then we get a nonnegative $(1,1)$ -form ω_0 whose $2n$ -th power is closed. Notice that $G : \Sigma_g \rightarrow \mathbb{CP}^1$ is a branched cover, therefore ω_0^{2n} , identified as a $(1,1)$ -form via a volume form on $\Sigma_g \times N$, is degenerate at those fibers of π over ramification points of G , only in the vertical direction.

To remedy this problem, for each ramification point $q \in \Sigma_g$, we construct a closed $(2n, 2n)$ -form λ_q on $\Sigma_g \times N$ as following. Let z be a local coordinate on Σ_g with $z(q) = 0$. Let f_t be a real smooth function on Σ_g such that f_t is given by

$$f_t(z) = (1 + |z|^2)^t$$

near q . Consider the closed real $(2n, 2n)$ -form

$$\lambda_q = i\partial\bar{\partial}(f_t(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)^{2n-1}).$$

Evaluate it at q , we see that

$$\lambda_q(q) = itdz \wedge d\bar{z} \wedge (\alpha\omega_I + \beta\omega_J + \gamma\omega_K)^{2n-1} + i\partial\bar{\partial}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)^{2n-1}.$$

Therefore for t big enough, $\lambda_q(q)$ identified as a $(1,1)$ -form, is strictly positive definite in the vertical direction.

Now let

$$\lambda_0 = \omega_0^{2n} + \sum_q c_q \lambda_q,$$

where c_q are positive constants. Our argument shows that if we choose c_q close to 0, then λ_0 is a closed positive $(2n, 2n)$ -form on $\Sigma_g \times N$, hence we prove our theorem.

It should be noticed that the only fact about G we used in the proof is that it is a branched covering. \square

Now let us assume that N is hyperkähler and we replace $G : \Sigma_g \rightarrow \mathbb{CP}^1$ by an arbitrary holomorphic map $h : Y \rightarrow \mathbb{CP}^1$. Let \tilde{Y} be the total space of the pullback fibration h^*Z . Our computation above implies that

$$K_{\tilde{Y}} \cong K_Y \otimes h^*\mathcal{O}(-2n),$$

where K_Y and $K_{\tilde{Y}}$ represent the canonical bundle of Y and \tilde{Y} respectively. Use an argument similar to [Tom15, Corollary 1], we actually have proved

Theorem 4.6. If $h : Y \rightarrow \mathbb{P}^1$ satisfies

$$(8) \quad K_Y \cong h^*\mathcal{O}(2n),$$

then $\tilde{Y} = h^*Z$ has trivial canonical bundle. As long as h is not a constant map, \tilde{Y} is non-Kähler.

Remark 4.7. A similar construction was used by LeBrun [LeB99] for different purposes.

If (8) is satisfied, then $L = h^*\mathcal{O}(n)$ is a square root of K_Y , which corresponds to a spin structure on Y according to Atiyah [Ati71]. L is known as a theta characteristic in the case that Y is a curve. The minimal surface Σ_g we considered is a special case of such construction with $n = 1$. For Y a curve and $n = 1$, such an h exists if and only if there is a theta characteristic L on Y such that $h^0(Y, L) \geq 2$, i.e., L is a vanishing theta characteristic.

We shall see that Theorem 4.6 is indeed a more general construction compared to Calabi-Gray. However, we do not have any natural metric on \tilde{Y} from this consideration.

Example 4.8. Let Y be a smooth $g = 3$ curve and set $n = 1$. It is well known, see [GH04, Section 4] for instance, that Y admits a vanishing theta characteristic if and only if Y is hyperelliptic. Hyperelliptic genus 3 curves have a moduli of complex dimension 5, while it seems that the largest family of minimal genus 3 curves in T^3 we know is of real dimension 5 constructed in [MI90, Theorem 7.1].

Example 4.9. For every hyperelliptic curves Y of genus $g \geq 3$, vanishing theta characteristics exist, so Theorem 4.6 can be used to construct non-Kähler Calabi-Yau 3-folds. However, it is a theorem of Meeks, see [MI90, Theorem 3.3], that if g is even, Y can not be minimally immersed in T^3 . From this we see that Theorem 4.6 yields examples cannot be covered by Calabi-Gray.

Actually, the set of genus g curves with a vanishing theta characteristic form a divisor in the moduli space of genus g curves. More refined results of this type can be found in [Har82] and [TiB87].

Example 4.10. If we allow Y to be of higher dimension and n to be greater than 1, then Theorem 4.6 can be used to construct lots more non-Kähler Calabi-Yau manifolds of higher dimension. For instance, we can take $Y \subset \mathbb{CP}^1 \times \mathbb{CP}^r$ to be a smooth hypersurface of bidegree $(2n + 2, r + 1)$, then (8) is satisfied, where h is the restriction of the projection to \mathbb{CP}^1 . There are also numerous examples of elliptic fibrations over \mathbb{CP}^1 without multiple fibers such that (8) holds. These examples lead to simply-connected non-Kähler Calabi-Yau's of complex dimension not less than 4.

Remark 4.11. It has been known for many years that the Iitaka conjecture fails for general compact complex manifolds, see [NU73, Remark 4] and [Mag12, Example 2]. It seems that all the counterexamples the author can find in literature involve non-Kähler manifolds with negative Kodaira dimension. On the other hand, the fibration $\pi : \tilde{Y} \rightarrow Y$ we constructed in Theorem 4.6 has the property that $\kappa(\pi_y) = \kappa(\tilde{Y}) = 0$ while $\kappa(Y) > 0$, hence

$$\kappa(\tilde{Y}) < \kappa(Y) + \kappa(\pi_y),$$

violating the assertion of Iitaka conjecture.

Part II

A Degenerate Solution to the Strominger System

In Part I we constructed various non-Kähler Calabi-Yau manifolds with balanced metric. They are natural testing ground for heterotic strings. In Part II of this paper, we will only consider the simplest case, i.e., $X = \Sigma_g \times T^4$, where Σ_g is a minimal genus g Riemann surface in T^3 and T^4 is the real 4-torus with standard hyperkähler structure. As we have seen in Section 2 that X is a non-Kähler Calabi-Yau 3-fold whose natural metric is specially balanced. We will try to solve Strominger system on X based on the idea from [FY08].

5 More Complex Geometry

Recall that the Strominger system

$$\begin{aligned} F \wedge \omega^2 &= 0, \quad F^{0,2} = F^{2,0} = 0 \\ \sqrt{-1} \partial \bar{\partial} \omega &= \frac{\alpha'}{4} (\text{Tr } R \wedge R - \text{Tr } F \wedge F) \\ d(\|\Omega\|_\omega \cdot \omega^2) &= 0 \end{aligned}$$

involves curvature forms F and R . In this work, they will be computed with respect to the Chern connection. To do that, it is convenient to work with a holomorphic frame.

Let e^4, e^5, e^6, e^7 be a set of parallel orthonormal 1-forms on T^4 and we will use the convention

$$\begin{aligned}\omega_I &= e^4 \wedge e^5 + e^6 \wedge e^7, \\ \omega_J &= e^4 \wedge e^6 - e^5 \wedge e^7, \\ \omega_K &= e^4 \wedge e^7 + e^5 \wedge e^6.\end{aligned}$$

In terms of this frame, it is straightforward to write down the holomorphic (3,0)-form $\Omega = \Omega_1 + i\Omega_2$ where

$$\begin{aligned}\Omega_1 &= e^1 \wedge \omega_I + e^2 \wedge \omega_J + e^3 \wedge \omega_K, \\ \Omega_2 &= (-\gamma e^2 + \beta e^3) \wedge \omega_I + (\gamma e^1 - \alpha e^3) \wedge \omega_J + (-\beta e^1 + \alpha e^2) \wedge \omega_K.\end{aligned}$$

Now we proceed to solve for a local holomorphic frame on (X, J_0) . It is easier to work with 1-forms. Consider a (1,0)-form θ of the form

$$\theta = Ldz + Ae^4 + Be^5 + Ce^6 + De^7,$$

where z is a local holomorphic coordinate on Σ_g while L, A, B, C, D are smooth functions to be determined.

As $J_0\theta = i\theta$, it follows that

$$\begin{aligned}iA &= \alpha B + \beta C + \gamma D, \\ iB &= -\alpha A + \gamma C - \beta D, \\ iC &= -\beta A - \gamma B + \alpha D, \\ iD &= -\gamma A + \beta B - \alpha C.\end{aligned}$$

Solve A and B from C, D , we get

$$\begin{aligned}(9) \quad A &= -\frac{\alpha\gamma + i\beta}{\beta^2 + \gamma^2}C + \frac{\alpha\beta - i\gamma}{\beta^2 + \gamma^2}D = -\kappa C + \sigma D, \\ B &= \frac{\alpha\beta - i\gamma}{\beta^2 + \gamma^2}C + \frac{\alpha\gamma + i\beta}{\beta^2 + \gamma^2}D = \sigma C + \kappa D,\end{aligned}$$

where

$$\kappa = \frac{\alpha\gamma + i\beta}{\beta^2 + \gamma^2} = \frac{i}{2} \left(\zeta + \frac{1}{\zeta} \right) \text{ and } \sigma = \frac{\alpha\beta - i\gamma}{\beta^2 + \gamma^2} = -\frac{1}{2} \left(\zeta - \frac{1}{\zeta} \right)$$

are holomorphic functions.

If θ is a holomorphic (1,0)-form, then

$$d\theta = dL \wedge dz + dA \wedge e^4 + dB \wedge e^5 + dC \wedge e^6 + dD \wedge e^7$$

is of type (2,0), which is equivalent to that

$$J_0(d\theta) = -d\theta.$$

As a consequence, we have

$$\begin{aligned}& (dA + \alpha J_0 dB + \beta J_0 dC + \gamma J_0 dD) \wedge e^4 + (dB - \alpha J_0 dA + \gamma J_0 dC - \beta J_0 dD) \wedge e^5 \\ & + (dC - \beta J_0 dA - \gamma J_0 dB + \alpha J_0 dD) \wedge e^6 + (dD - \gamma J_0 dA + \beta J_0 dB - \alpha J_0 dC) \wedge e^7 + 2\bar{\partial}L \wedge dz \\ & = 0.\end{aligned}$$

Plug in (9), we can compute

$$\begin{aligned} dA + \alpha J_0 dB + \beta J_0 dC + \gamma J_0 dD &= -2\kappa \bar{\partial}C + 2\sigma \bar{\partial}D + C(i\alpha \partial\sigma - \partial\kappa) + D(\partial\sigma + i\alpha \partial\kappa), \\ dB - \alpha J_0 dA + \gamma J_0 dC - \beta J_0 dD &= 2\sigma \bar{\partial}C + 2\kappa \bar{\partial}D + C(\partial\sigma + i\alpha \partial\kappa) - D(i\alpha \partial\sigma - \partial\kappa), \\ dC - \beta J_0 dA - \gamma J_0 dB + \alpha J_0 dD &= 2\bar{\partial}C + iC(\beta \partial\kappa - \gamma \partial\sigma) - iD(\beta \partial\sigma + \gamma \partial\kappa), \\ dD - \gamma J_0 dA + \beta J_0 dB - \alpha J_0 dC &= 2\bar{\partial}D + iC(\beta \partial\sigma + \gamma \partial\kappa) - iD(\gamma \partial\sigma - \beta \partial\kappa). \end{aligned}$$

Therefore

$$\begin{aligned} &2\bar{\partial}L \wedge dz + 2\bar{\partial}C \wedge (-\kappa e^4 + \sigma e^5 + e^6) + 2\bar{\partial}D \wedge (\sigma e^4 + \kappa e^5 + e^7) \\ &+ (C\partial\sigma + D\partial\kappa) \wedge (i\alpha e^4 + e^5 - i\gamma e^6 + i\beta e^7) + (C\partial\kappa - D\partial\sigma) \wedge (-e^4 + i\alpha e^5 + i\beta e^6 + i\gamma e^7) \\ &= 0. \end{aligned}$$

Each term in the above equation is a (1,1) form. Notice that

$$\{dz, -\kappa e^4 + \sigma e^5 + e^6, \sigma e^4 + \kappa e^5 + e^7\}$$

form a basis for (1,0)-forms, so we deduce that $\bar{\partial}C = \bar{\partial}D = 0$ and

$$2\bar{\partial}L = (C\sigma_z + D\kappa_z)(i\alpha e^4 + e^5 - i\gamma e^6 + i\beta e^7) + (C\kappa_z - D\sigma_z)(-e^4 + i\alpha e^5 + i\beta e^6 + i\gamma e^7),$$

which is always locally solvable since the right hand side is $\bar{\partial}$ -closed.

Therefore we conclude that

$$\{dz, L_1 dz - \kappa e^4 + \sigma e^5 + e^6, L_2 dz + \sigma e^4 + \kappa e^5 + e^7\}$$

is a local holomorphic frame of $T_{\mathbb{C}}^*X$, where L_1 and L_2 are functions satisfying

$$\begin{aligned} (10) \quad 2\bar{\partial}L_1 &= \sigma_z(e^5 + iJ_0 e^5) - \kappa_z(e^4 + iJ_0 e^4) = \frac{2i\alpha_z}{\beta^2 + \gamma^2}(e^7 + iJ_0 e^7), \\ 2\bar{\partial}L_2 &= \kappa_z(e^5 + iJ_0 e^5) + \sigma_z(e^4 + iJ_0 e^4) = -\frac{2i\alpha_z}{\beta^2 + \gamma^2}(e^6 + iJ_0 e^6). \end{aligned}$$

After taking dual basis and rescaling, we obtain a holomorphic frame of $T_{\mathbb{C}}X$ as follows

$$\begin{aligned} (11) \quad V_1 &= i\beta e_4 + i\gamma e_5 + e_6 - i\alpha e_7 = e_6 - iJ_0 e_6, \\ V_2 &= i\gamma e_4 - i\beta e_5 + i\alpha e_6 + e_7 = e_7 - iJ_0 e_7, \\ V_0 &= 2\frac{\partial}{\partial z} - L_1 V_1 - L_2 V_2. \end{aligned}$$

Observe that V_1 and V_2 are globally defined and nowhere vanishing. Similarly $e_4 - iJ_0 e_4$ and $e_5 - iJ_0 e_5$ are nowhere vanishing holomorphic vector fields on X .

At point where $(\alpha, \beta, \gamma) = (1, 0, 0)$, we have $V_1 + iV_2 = 0$. Similarly at point where $(\alpha, \beta, \gamma) = (-1, 0, 0)$, we see $V_1 - iV_2 = 0$. As the Gauss map is surjective, we conclude that as holomorphic vector fields, both $V_1 + iV_2$ and $V_1 - iV_2$ have zeroes.

In [LS94, Theorem 1], LeBrun and Simanca proved that on a compact Kähler manifold, the set of holomorphic vector fields with zeroes is actually a vector space. Hence we obtain a different proof that X is non-Kähler. In fact, we can prove a little more:

Proposition 5.1. X does not satisfy the $\partial\bar{\partial}$ -lemma. As a corollary, X is not of Fujiki class \mathcal{C} .

Proof. Let ξ be a holomorphic (1,0)-form on X . Since its pairing with V_1 and V_2 are constants, one can easily deduce that ξ must be pullback of a holomorphic (1,0)-form from Σ_g . Hence $h^0(X, \Omega^1) = h^{1,0}(X) = g$. On the other hand, $b_1(X) = 2g + 4 > 2g = 2h^{1,0}(X)$, therefore the $\partial\bar{\partial}$ -lemma fails. \square

Jost and Yau [JY93] introduced the concept of astheno-Kähler metric, which means an Hermitian metric ω' satisfying $\partial\bar{\partial}(\omega'^{n-2}) = 0$. In complex dimension 3, it reads $\partial\bar{\partial}\omega' = 0$, which coincides with the notion of SKT (strong Kähler with torsion, also known as pluriclosed) metric. An import result is the following obstruction of astheno-Kähler metric found by Jost-Yau:

Theorem 5.2. (Jost-Yau [JY93, Lemma 6])

Let M be a compact astheno-Kähler manifold, then every holomorphic 1-form on M is closed.

We shall point out that this obstruction is not enough. There exist a nilmanifold satisfying the condition of Theorem 5.2 which does not support any astheno-Kähler metric, as observed in [FT11, Example 2.3]. This phenomenon will be manifested below as well.

A folklore conjecture says if a compact complex manifold admits both balanced and astheno-Kähler metrics or both balanced and SKT metrics (a priori they are different), then it must be Kähler. The SKT version of this conjecture has been solved in a few cases, including connected sums of $S^3 \times S^3$ [FLY12], twistor spaces of anti-self-dual 4-manifolds [Ver14], manifolds of Fujiki class \mathcal{C} [Chi14] and nilmanifolds [FV15a, FV15b].

To verify this conjecture for our X , we prove

Theorem 5.3. X is not astheno-Kähler/SKT.

Proof. From Proposition 5.1, we know that all the holomorphic 1-forms on X are d-closed, therefore we cannot apply Theorem 5.2 directly. In addition, we know that X is not of Fujiki class \mathcal{C} from Proposition 5.1, therefore this theorem is not covered by Chiose's result [Chi14].

Let $\rho^j = e^j - iJ_0e^j$ for $j = 4, 5, 6, 7$. Clearly they are (1,0)-forms on X . Observe that

$$d\rho^j = -id(J_0e^j)$$

is purely imaginary. On the other hand, $d\rho^j$ is of type (2,0)+(1,1), therefore we conclude that $\partial\rho^j = 0$ and

$$\bar{\partial}\rho^j = -id(J_0e^j).$$

Assume that X admits an astheno-Kähler metric ω' , then by integration by part, we have

$$\int_X (d(J_0\rho^j))^2 \wedge \omega' = \int_X \bar{\partial}\rho^j \wedge \partial\bar{\rho}^j \wedge \omega' = \int_X \rho^j \wedge \bar{\rho}^j \wedge \partial\bar{\partial}\omega' = 0.$$

On the other hand, explicit calculation shows that

$$\sum_{j=4}^7 (d(J_0\rho^j))^2 = -4d\beta \wedge d\gamma \wedge \omega_I - 4d\gamma \wedge d\alpha \wedge \omega_J - 4d\alpha \wedge d\beta \wedge \omega_K.$$

Observe that

$$\frac{d\beta \wedge d\gamma}{\alpha} = \frac{d\gamma \wedge d\alpha}{\beta} = \frac{d\alpha \wedge d\beta}{\gamma} = \frac{id\zeta \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2} = G^*\omega_{\mathbb{CP}^1}$$

is the pullback of the Fubini-Study metric by the Gauss map G . Therefore we have

$$0 = \sum_{j=4}^7 \int_X (d(J_0\rho^j))^2 \wedge \omega' = -4 \int_X G^*\omega_{\mathbb{CP}^1} \wedge (\alpha\omega_I + \beta\omega_J + \gamma\omega_K) \wedge \omega'.$$

This is in contradiction with the positivity of ω' . □

Remark 5.4. The result of Theorem 5.3 answers a question of Fu-Wang-Wu [FWW13, Section 1]. On the contrary, there exists 1-Gauduchon metrics on X , namely Hermitian metric ω' such that

$$\partial\bar{\partial}\omega' \wedge \omega' = 0.$$

This is a theorem of Fu-Wang-Wu [FWW13, Corollary 20].

Remark 5.5. Extracting from the above argument, we can define a map

$$u : H_{\text{dR}}^1(X; \mathbb{R}) \rightarrow H_{\text{BC}}^{1,1}(X; \mathbb{R})$$

on any compact complex manifold (X, J) , where J is the complex structure. The map u is defined to be

$$u : [\rho] \mapsto [d(J\rho)],$$

where H_{dR}^* and H_{BC}^* denote the de Rham cohomology and Bott-Chern cohomology respectively. If X satisfies the $\partial\bar{\partial}$ -lemma, then u is identically zero. In general, u detects the failure of $\partial\bar{\partial}$ -lemma. This map u is essentially related to the isomorphism

$$\frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{R})} \cong \frac{\{\text{d-exact real } (1,1)\text{-forms}\}}{\{i\partial\bar{\partial}\psi, \psi \in C^\infty(X, \mathbb{R})\}},$$

which is discussed in the proof of [Tos15, Proposition 1.6].

6 Solving Strominger System

6.1 Conformally Balanced Equation

Recall from Section 2 that the naturally induced metric (5)

$$\omega_0 = \omega + \alpha\omega_I + \beta\omega_J + \gamma\omega_K$$

is special balanced, therefore it solves (3). However, this metric is too restrictive for practical use. So we introduce a smooth function f on Σ_g and cook up a new metric

$$\omega_f = e^{2f}\omega + e^f(\alpha\omega_I + \beta\omega_J + \gamma\omega_K).$$

Obviously we have

$$\|\Omega\|_{\omega_f} = e^{-2f}\|\Omega\|_{\omega_0}$$

and

$$\omega_f^2 = 2e^{3f}\omega \wedge (\alpha\omega_I + \beta\omega_J + \gamma\omega_K) + 2e^{2f}e^4 \wedge e^5 \wedge e^6 \wedge e^7.$$

It follows that ω_f also solves the conformally balanced equation

$$d(\|\Omega\|_{\omega_f}\omega_f^2) = 0.$$

6.2 Curvature Computation

As we have worked out a local holomorphic frame in Section 5, we are able to compute the term $\text{Tr}(R_f \wedge R_f)$ in (2) with respect to the Chern connection associated to ω_f .

With respect to the local holomorphic frame $\{V_0, V_1, V_2\}$, the metric ω_f is given by the matrix

$$H = 2e^f \begin{pmatrix} e^f\lambda + |L_1|^2 + |L_2|^2 - i\alpha(L_1\bar{L}_2 - L_2\bar{L}_1) & -L_1 - i\alpha L_2 & -L_2 + i\alpha L_1 \\ -\bar{L}_1 + i\alpha\bar{L}_2 & 1 & -i\alpha \\ -\bar{L}_2 - i\alpha\bar{L}_1 & i\alpha & 1 \end{pmatrix}.$$

The inverse matrix can be computed accordingly

$$H^{-1} = \frac{1}{2e^{2f}\lambda} \begin{pmatrix} 1 & L_1 & L_2 \\ \bar{L}_1 & |L_1|^2 & L_2\bar{L}_1 \\ \bar{L}_2 & L_1\bar{L}_2 & |L_2|^2 \end{pmatrix} + \frac{1}{2e^f(1-\alpha^2)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i\alpha \\ 0 & -i\alpha & 1 \end{pmatrix}.$$

Let

$$p = e^{2f}\lambda,$$

$$R = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix},$$

and set

$$U = \begin{pmatrix} -L_1 & -L_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -L \\ \text{id} \end{pmatrix},$$

$$S = e^f \begin{pmatrix} 1 & -i\alpha \\ i\alpha & 1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 \\ \bar{L}_1 \\ \bar{L}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \bar{L}^T \end{pmatrix}.$$

We can express H and H^{-1} as

$$H = 2pR + 2US\bar{U}^T,$$

$$H^{-1} = \frac{1}{2p}V\bar{V}^T + \frac{1}{2} \begin{pmatrix} 0 & \\ & S^{-1} \end{pmatrix}.$$

Direct computation shows that

$$\bar{H}^{-1}\partial\bar{H} = (p^{-1}\partial p) \begin{pmatrix} 1 & 0 \\ L^T & 0 \end{pmatrix} + p^{-1} \begin{pmatrix} \partial\bar{L} \\ L^T\partial\bar{L} \end{pmatrix} (\bar{S}L^T \quad -\bar{S}) + \begin{pmatrix} 0 \\ \bar{S}^{-1}\partial\bar{S} \end{pmatrix} (-L^T \quad \text{id}) - \begin{pmatrix} 0 & 0 \\ \partial L^T & 0 \end{pmatrix}.$$

As a consequence, we have

$$\begin{aligned} R_f = \bar{\partial}(\bar{H}^{-1}\partial\bar{H}) &= (\bar{\partial}\partial\log p) \begin{pmatrix} 1 & 0 \\ L^T & 0 \end{pmatrix} - \partial\log p \wedge \begin{pmatrix} 0 & 0 \\ \bar{\partial}L^T & 0 \end{pmatrix} - \frac{\bar{\partial}p}{p^2} \begin{pmatrix} \partial\bar{L} \\ L^T\partial\bar{L} \end{pmatrix} (\bar{S}L^T \quad -\bar{S}) \\ &+ p^{-1} \begin{pmatrix} \bar{\partial}\partial\bar{L} \\ \bar{\partial}(L^T\partial\bar{L}) \end{pmatrix} (\bar{S}L^T \quad -\bar{S}) - p^{-1} \begin{pmatrix} \partial\bar{L} \\ L^T\partial\bar{L} \end{pmatrix} (\bar{\partial}(\bar{S}L^T) \quad -\bar{\partial}\bar{S}) + \begin{pmatrix} 0 \\ \bar{\partial}(\bar{S}^{-1}\partial\bar{S}) \end{pmatrix} (-L^T \quad \text{id}) \\ &+ \begin{pmatrix} 0 \\ \bar{S}^{-1}\partial\bar{S} \end{pmatrix} (\bar{\partial}L^T \quad 0) - \begin{pmatrix} 0 & 0 \\ \bar{\partial}\partial L^T & 0 \end{pmatrix}. \end{aligned}$$

From this we can see immediately that

$$\text{Tr}(R_f) = 4\bar{\partial}\partial f.$$

A even more complicated calculation reveals that

$$\begin{aligned} \text{Tr}(R_f \wedge R_f) &= 2 \left[-\frac{\bar{\partial}\partial\log p}{p} \partial\bar{L} \cdot \bar{S} \cdot \bar{\partial}L^T - \frac{\partial\log p \wedge \bar{\partial}p}{p^2} \partial\bar{L} \cdot \bar{S} \cdot \bar{\partial}L^T + \frac{\partial\log p}{p} \bar{\partial}\partial\bar{L} \cdot \bar{S} \cdot \bar{\partial}L^T \right. \\ &\quad - \frac{\partial\log p}{p} \partial\bar{L} \cdot \bar{\partial}\bar{S} \cdot \bar{\partial}L^T + \frac{\bar{\partial}p}{p^2} \partial\bar{L} \cdot \partial\bar{S} \cdot \bar{\partial}L^T - \frac{\bar{\partial}p}{p^2} \partial\bar{L} \cdot \bar{S} \cdot \bar{\partial}\partial L^T + \frac{1}{p} \partial\bar{L} \cdot \bar{S} \bar{\partial}(\bar{S}^{-1}\partial\bar{S}) \bar{\partial}L^T \\ &\quad \left. - \frac{1}{p} \bar{\partial}\partial\bar{L} \cdot \partial\bar{S} \cdot \bar{\partial}L^T + \frac{1}{p} \bar{\partial}\partial\bar{L} \cdot \bar{S} \cdot \bar{\partial}\partial L^T + \frac{1}{p} \partial\bar{L} \cdot \bar{\partial}\bar{S} \cdot \bar{S}^{-1}\partial\bar{S} \cdot \bar{\partial}L^T - \frac{1}{p} \partial\bar{L} \cdot \bar{\partial}\bar{S} \cdot \bar{\partial}\partial L^T \right]. \end{aligned}$$

Let $W = \partial\bar{L} \cdot \bar{S} \bar{\partial}L^T$. After a recombination of terms, we get a very simple expression

$$\begin{aligned} \text{Tr}(R_f \wedge R_f) &= -\frac{2}{p} [(\bar{\partial}\partial\log p + \partial\log p \wedge \bar{\partial}\log p)W - \partial\log p \wedge \bar{\partial}W + \bar{\partial}\log p \wedge \partial W - \bar{\partial}\partial W] \\ &= 2\bar{\partial}\partial \left(\frac{W}{p} \right). \end{aligned}$$

Recall that $\bar{\partial}L$ can be read off from (10), hence we are able to calculate this term explicitly as

$$\frac{W}{p} = -\frac{2i}{e^f \lambda} \frac{|\alpha_z|^2}{\beta^2 + \gamma^2} (\alpha\omega_I + \beta\omega_J + \gamma\omega_K) = -\frac{i}{4e^f} \|dG\|^2 (\alpha\omega_I + \beta\omega_J + \gamma\omega_K),$$

where $G : \Sigma_g \rightarrow \mathbb{CP}^1$ is the Gauss map. Clearly it is globally defined.

6.3 Solving the Whole System

A crucial consequence of the lengthy calculation above is that $\text{Tr}(R_f \wedge R_f)$ is $\partial\bar{\partial}$ -exact. Therefore we can simply take $F \equiv 0$ to solve the Hermitian-Yang-Mills (1) without violating the cohomological restriction in (2).

We also observe that

$$i\partial\bar{\partial}\omega_f = i\partial\bar{\partial}(e^f(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)).$$

Therefore by equating

$$e^{2f} = \frac{\alpha'}{8} \|dG\|^2,$$

we solve the whole Strominger system with $F \equiv 0$.

Unfortunately $\|dG\|^2$ vanishes at the ramification points, at which f goes to $-\infty$, thus the metric ω_f is degenerate at the fibers of π over these ramification points. The ramification locus is unavoidable and related issues will be addressed in next section. What we get is a degenerate solution to the Strominger system.

7 The Geometry of Degenerate Locus and Further Discussions

A simple application of Riemann-Hurwitz formula shows that G has $4(g-1)$ ramification points counted with multiplicity. These are exactly the zeroes of Gauss curvature of Σ_g . It is a very interesting question if we can find a nontrivial lower bound for number of zeroes of Gauss curvature when counting without multiplicity.

Now let us consider the case $\Sigma_g \looparrowright T^3$ is a hyperelliptic minimal surface of genus g . Meeks showed that g must be odd. More importantly, Meeks [MI90, Proposition 3.1, Theorem 3.2] observed that the hyperelliptic automorphism of Σ_g is an isometry that is induced by an inversion symmetry in T^3 through any hyperelliptic point. Furthermore, after a translation in T^3 one can manage to locate every hyperelliptic point of Σ_g inside the set of order 2 points in T^3 . In an ideal case with $g=3$, the 8 hyperelliptic points will be exactly the 8 order 2 points in T^3 .

From this we see that $\|dG\|^2$ is invariant under the hyperelliptic automorphism τ , hence our solution to the anomaly cancellation equation (2) descends to $X/\langle\tau\rangle$, which is a T^4 fibration over \mathbb{CP}^1 with orbifold singularities. However, we notice that $\tau^*\Omega = -\Omega$, therefore what we really have on $X/\langle\tau\rangle$ is a solution to the “twisted” Strominger system.

In summary, on non-Kähler Calabi-Yau 3-folds of the type $\Sigma_g \times T^4$, we are able to write down explicit solutions to the whole Strominger system with degeneracies. A curious feature is that infinitely many topological types occur in these models. Nevertheless, there are many interesting questions left unanswered. For example, one may ask if there are any interesting physics behind the degenerate locus. If not, is it possible to apply a perturbation argument to get rid of the degeneracies? Or maybe more importantly, can one add in nontrivial $\text{Tr}(F \wedge F)$ -term to fix these degeneracies? Preferably F may come from anti-self-dual instantons on T^4 . By the famous Atiyah-Ward correspondence [AHS78], the pullback of such instantons to X have holomorphic structures satisfying the Hermitian-Yang-Mills Equation (1) automatically for our ansatz ω_f . However, the anomaly cancellation (2) still awaits to be handled.

On the other hand, since the degeneracies concentrate on the fibers over ramification points, it is natural to consider the Strominger system on the twistor space instead of on the pull-back space X . However, a twistor space can never have trivial canonical bundle, therefore further modifications are needed. This idea will be developed in [Fei15a] where local models of torsional heterotic strings are described.

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